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ON THE POINCARÉ AND CHETAYEV EQUATIONS†

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The Lie group of virtual displacement operators in Rodrigues–Hamilton parameters is constructed and equations of motion are derived for a heavy rigid body with one fixed point. It is shown that the addition (subtraction) of a term of the form df/dt, $f(t,x) \in C^2$, to (from) the generalized Lagrangian $L^*(t, x, \eta)$ does not affect the form of the Poincaré and Chetayev equations. These equations can also be used to describe the relative motion of a holonomic system relative to a moving system of coördinates. Hamel's equations in non-linear quasi-coordinates are derived without using the transitivity equations, are compared with the generalized Poincaré equations and are transformed to Chetayev canonical form. © 1998 Elsevier Science Ltd. All rights reserved.

The equations of Poincaré [1] and Chetayev [2], based on the application of Lie groups of virtual displacement operators in dynamics, and their theory have been extended [3, 4] to closed systems of operators. The generalized equations are the general equations of classical mechanics, which include all known equations of motion (without factors for constraints) of holonomic and non-holonomic mechanical systems, among them also equations in linear quasi-velocities. The form of the equations—the same in independent and in dependent coordinates—depends on the choice of the parameters of the actual displacements, whose number equals that of the degrees of freedom.

In this paper, the equations of motion, in the Poincaré and Chetayev forms, will be used to describe the motion of a rigid body in Rodrigues–Hamilton parameters [5], as well as relative motion relative to a moving system of coordinates [6]. Finally, we will present a derivation of the Hamel equations in non-linear quasi-velocities and compare them with the generalized Poincaré and Chetayev equations.

1. THE EQUATIONS OF MOTION OF A RIGID BODY IN RODRIGUES–HAMILTON PARAMETERS

In some problems of rigid body dynamics and in the theory of gyroscopic systems it is sometimes preferable to use Rodrigues–Hamilton parameters [5], which are dependent variables but, unlike the Euler angles, have no degeneracies.

Consider a heavy rigid body with one fixed point O, which we take to be the origin of a fixed system of coordinates axes $O\xi\eta\zeta$, with the $O\zeta$ axis pointing vertically upward, and a moving system of coordinates $Oxyz$ whose axes coincide with the principal axes of inertia of the body about the point O.

As defining coordinates we take the Rodrigues–Hamilton parameters λ_s ($s = 0, 1, 2, 3$), which satisfy the relation [6]

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \tag{1.1}$$

The projections p, q, r of the instantaneous velocity of the body on the x, y, z axes are expressed in terms of λ_s, $\dot{\lambda}_s$ as follows:

$$p = 2(\lambda_0\dot{\lambda}_1 - \lambda_1\dot{\lambda}_0 + \lambda_3\dot{\lambda}_2 - \lambda_2\dot{\lambda}_3) \quad (pqr, 1\ 2\ 3) \tag{1.2}$$

(here and below, it is understood that the unwritten relationships are obtained by the cyclic permutation indicated in parentheses).

Equations (1.2) and the relation

$$\lambda_0\dot{\lambda}_0 + \lambda_1\dot{\lambda}_1 + \lambda_2\dot{\lambda}_2 + \lambda_3\dot{\lambda}_3 = 0$$

imply the equations

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$$2\dot{\lambda}_0 = -(\lambda_1 p + \lambda_2 q + \lambda_3 r)$$

$$2\dot{\lambda}_1 = \lambda_0 p - \lambda_3 q + \lambda_2 r \quad (pqr,\ 1\ 2\ 3) \tag{1.3}$$

As Poincaré parameters of the actual displacements of the body η_s ($s = 1, 2, 3$) we take the quantities p, q and r, respectively. Using the expression for the time derivative of a function $f(t, \lambda_0, \lambda_1, \lambda_2, \lambda_3) \in C^2$

$$\dot{f} = \frac{\partial f}{\partial t} + \sum_{i=0}^{3} \frac{\partial f}{\partial \lambda_i}\dot{\lambda}_i = \frac{\partial f}{\partial t} + \sum_{s=1}^{3} \eta_s X_s f$$

we find the intransitive group of actual displacement operators $\partial/\partial t, X_s$ ($s = 1, 2, 3$), where the virtual displacement operators are

$$X_1 f = -\frac{1}{2}\left(\lambda_1 \frac{\partial f}{\partial \lambda_0} - \lambda_0 \frac{\partial f}{\partial \lambda_1} - \lambda_3 \frac{\partial f}{\partial \lambda_2} + \lambda_2 \frac{\partial f}{\partial \lambda_3}\right) \quad (1\ 2\ 3) \tag{1.4}$$

with commutators

$$[X_1, X_2]f = X_3 f \quad (1\ 2\ 3)$$

and the operator $\partial/\partial t$ commutes with the operators X_s. Consequently, the non-vanishing structure constants of the Lie group are

$$c_{12}^3 = c_{23}^1 = c_{31}^2 = 1, \quad c_{21}^3 = c_{32}^1 = c_{13}^2 = -1$$

The generalized Lagrangian is

$$L^*(\lambda_0, \lambda_1, \lambda_2, \lambda_3, p, q, r) = \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2) - \mathrm{Mg}[2(\lambda_1\lambda_3 - \lambda_0\lambda_2)x_0 +$$

$$+2(\lambda_0\lambda_1 + \lambda_2\lambda_3)y_0 + (\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2)z_0] \tag{1.5}$$

where A, B and C are the principal moments of inertia about O, Mg is the weight of the body, and x_0, y_0 and z_0 are the coordinates of its centre of gravity.

The Poincaré equations of motion [1] of a heavy rigid body with one fixed point take the form of the dynamical Euler equations

$$\begin{aligned} A\dot{p} - (B - C)qr &= \mathrm{Mg}[2(\lambda_0\lambda_1 + \lambda_2\lambda_3)z_0 - (\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2)y_0] \\ B\dot{q} - (C - A)rp &= \mathrm{Mg}[(\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2)x_0 - 2(\lambda_1\lambda_3 - \lambda_0\lambda_2)z_0] \\ C\dot{r} - (A - B)pq &= 2\mathrm{Mg}[(\lambda_1\lambda_3 - \lambda_0\lambda_2)y_0 - (\lambda_0\lambda_1 + \lambda_2\lambda_3)x_0] \end{aligned} \tag{1.6}$$

whose right-hand sides are expressed in terms of the Rodrigues–Hamilton parameters.

Equations (1.6) and (1.3) constitute a simultaneous system of seven first-order differential equations each with the same number of unknowns p, q, r, λ_s ($s = 0, 1, 2, 3$), the last four of which satisfy relation (1.1). These equations are analogous to the classical Euler equations in variables $p, q, r, \theta, \varphi, \psi$ and in the general case have, besides the integral (1.1), energy and area integrals; if the system is dynamically symmetric, that is, $A = B_2\, x_0 = y_0 = 0$, they have another integral: the constant projection of the instantaneous angular velocity onto the axis of dynamic symmetry.

Compared with Koshlyakov's system (1.7.6) of four second-order equations [5] each system (1.6), (1.3) is of one less order.

We now transform Eqs (1.6) and (1.3) to the form of the Chetayev canonical equations [2]. Introducing the variables

$$y_1 = \frac{\partial L^*}{\partial p} = Ap, \quad y_2 = \frac{\partial L^*}{\partial q} = Bq, \quad y_3 = \frac{\partial L^*}{\partial r} = Cr$$

and the generalized Hamiltonian

$$H^*(\lambda_0,\lambda_1,\lambda_2,\lambda_3,y_1,y_2,y_3)=\frac{1}{2}\left(\frac{y_1^2}{A}+\frac{y_2^2}{B}+\frac{y_3^2}{C}\right)+\mathrm{Mg}[2(\lambda_1\lambda_3-\lambda_0\lambda_2)x_0+$$
$$+2(\lambda_0\lambda_1+\lambda_2\lambda_3)y_0+(\lambda_0^2+\lambda_3^2-\lambda_1^2-\lambda_2^2)z_0]$$

we obtain the canonical Chetayev equations [2] in the Rodrigues–Hamilton parameters

$$\begin{aligned}
&\dot y_1+\left(\frac{1}{B}-\frac{1}{C}\right)y_2y_3=\mathrm{Mg}[2(\lambda_0\lambda_1+\lambda_2\lambda_3)z_0-(\lambda_0^2+\lambda_3^2-\lambda_1^2-\lambda_2^2)y_0]\\
&\dot y_2+\left(\frac{1}{C}-\frac{1}{A}\right)y_3y_1=\mathrm{Mg}[(\lambda_0^2+\lambda_3^2-\lambda_1^2-\lambda_2^2)x_0-2(\lambda_1\lambda_3-\lambda_0\lambda_2)z_0]\\
&\dot y_3+\left(\frac{1}{A}-\frac{1}{B}\right)y_1y_2=2\mathrm{Mg}[2(\lambda_1\lambda_3-\lambda_0\lambda_2)y_0-(\lambda_0\lambda_1+\lambda_2\lambda_3)x_0]\\
&2\dot\lambda_0=-\left(\lambda_1\frac{y_1}{A}+\lambda_2\frac{y_2}{B}+\lambda_3\frac{y_3}{C}\right)\\
&2\dot\lambda_1=\lambda_0\frac{y_1}{A}-\lambda_3\frac{y_2}{B}+\lambda_2\frac{y_3}{C}\quad (ABC,\ 1\ 2\ 3)
\end{aligned}\tag{1.7}$$

Chetayev's theory of integration [2] is applicable to Eqs (1.7).

Note that Eqs (1.7) have the form of Hamilton equations in the non-canonical variables y_s, λ_i [3]

$$\dot y_s=(y_s,H^*),\quad \dot\lambda_i=(\lambda_i,H^*),\quad s=1,2,3;\quad i=0,1,2,3$$

where the generalized Poisson bracket of two smooth functions $f(x,y)$ and $\varphi(x,y)$ is defined for a system with k degrees of freedom by

$$(f,\varphi)=\sum_{s=1}^{k}\left(\frac{\partial\varphi}{\partial y_s}X_sf-\frac{\partial f}{\partial y_s}X_s\varphi\right)+\sum_{s,r,m=1}^{k}c_{sr}^m\frac{\partial f}{\partial y_r}\frac{\partial\varphi}{\partial y_s}y_m$$

2. THE EQUATIONS OF RELATIVE MOTION

We will show that the generalized Poincaré equations for a holonomic mechanical system with k degrees of freedom

$$\frac{d}{dt}\frac{\partial L^*}{\partial\eta_s}=(c_{rs}^m\eta_r+c_{0s}^m)\frac{\partial L^*}{\partial\eta_m}+X_sL^*,\quad m,r,s=1,\dots,k\tag{2.1}$$

(with summation over repeated indices), with generalized Lagrangian of the form

$$L^*(t,x,\eta)=\Lambda(t,x,\eta)+\frac{df}{dt},\quad f(t,x)\in C^2\tag{2.2}$$

are equivalent to the equations

$$\frac{d}{dt}\frac{\partial\Lambda}{\partial\eta_s}=(c_{rs}^m\eta_r+c_{0s}^m)\frac{\partial\Lambda}{\partial\eta_m}+X_s\Lambda,\quad m,r,s=1,\dots,k\tag{2.3}$$

where $x_i (i=1,\dots,n\geqslant k)$ are the defining coordinates of the system.

Indeed, the term df/dt, which is expressible as

$$df/dt=X_0f+\eta_sX_sf$$

on the left of Eq. (2.1) leads to the expression

$$dX_sf/dt=X_0X_sf+\eta_rX_rX_sf=X_sX_0f+c_{0s}^mX_mf+\eta_r(X_sX_rf+c_{rs}^mX_mf),\quad r,s=1,\dots,k$$

which is just the sum of the terms of the right of Eq. (2.1) that originate from df/dt.

Consequently, adding (subtracting) the term df/dt to (from) the function $L^*(t, x, \eta)$ does not affect the Poincaré equations, as is the case for the Lagrange equations in independent coordinates q_i [7].

On the assumption that $\| \partial^2 L^*/\partial\eta_r\partial\eta_s \| \neq 0$ $(r, s = 1, \ldots, k)$, we apply Legendre transformations

$$y_s = \partial L^* / \partial\eta_s, \quad s = 1, \ldots, k, \quad H^*(t, x, y) = y_s\eta_s - L^* \tag{2.4}$$

to Eqs (2.1) and

$$Y_s = \partial\Lambda / \partial\eta_s, \quad s = 1, \ldots, k, \quad K(t, x, Y) = Y_s\eta_s - \Lambda \tag{2.5}$$

to Eqs (2.3). As a result we obtain the canonical Chetayev equations

$$\frac{dy_s}{dt} = \left(c_{rs}^m \frac{\partial H^*}{\partial y_r} + c_{0s}^m \right) y_m - X_s H^*, \quad \eta_s = \frac{\partial H^*}{\partial y_s}; \quad m, r, s = 1, \ldots, k \tag{2.6}$$

and

$$\frac{dy_s}{dt} = \left(c_{rs}^m \frac{\partial K}{\partial y_r} + c_{0s}^m \right) y_m - X_s K, \quad \eta_s = \frac{\partial K}{\partial y_s}; \quad m, r, s = 1, \ldots, k \tag{2.7}$$

which are equivalent to Eqs (2.1) and (2.3), respectively. Taking (2.4) and (2.5) into account, we conclude that

$$y_s = Y_s + X_s f, \quad H^*(t, x, y) = K(t, x, Y) - X_0 f$$

and using these equalities we find relations

$$\frac{\partial H^*}{\partial y_s} = \frac{\partial K}{\partial Y_s}, \quad s = 1, \ldots, k; \quad \frac{dy_s}{dt} = \frac{dY_s}{dt} + X_0 X_s f$$

from which it follows that Eqs (2.6) and (2.7) are equivalent. In relation to the last equality, we observe that in the canonical equations the variables x_i and y_s $(i = 1, \ldots, n;\ s = 1, \ldots, k)$ are considered independent.

An example of a function of the form (2.2) is the Lagrangian

$$L^*(t, x, \eta) = T^*(t, x, \eta) + U(t, x)$$

for the problem of relative motion of the system in a moving system of coordinates whose motion is known, where the defining coordinates x_i define the position of the system relative to the moving axes.

The motion of the moving system of coordinates $Oxyz$ is characterized by two vectors: the absolute velocity $\mathbf{v}_0(t)$ of the origin O and the absolute instantaneous angular velocity $\omega(t)$ of rotation about O. The kinetic energy of absolute motion of the system [6] is

$$T_a = T_r + \boldsymbol{\omega} \cdot G_r + \boldsymbol{\omega} \cdot \boldsymbol{\theta}^\circ \cdot \boldsymbol{\omega} / 2 + M\mathbf{v}_0 \cdot (\dot{\mathbf{r}}_c + \boldsymbol{\omega} \times \mathbf{r}_c) + Mv_0^2 / 2 \tag{2.8}$$

where $T_r = m_\nu \dot{\mathbf{r}}_\nu^2/2$ and $\mathbf{G}_r = \mathbf{r}_\nu \times m_\nu \dot{\mathbf{r}}_\nu$ $(\nu = 1, \ldots, N)$ are the kinetic energy and the relative angular momentum, $\mathbf{r}_\nu = \mathbf{r}_\nu(x_1, \ldots, x_n)$ is a radius vector with origin at O, $\dot{\mathbf{r}}_\nu$ is the relative velocity of a point mass m_ν, $M = \Sigma_\nu m_\nu$ is the mass, $\mathbf{r}_c$ is the radius vector of the centre of mass and $\boldsymbol{\theta}^\circ$ is the inertia tensor of the system about O. The time derivatives $\dot{\mathbf{r}}_\nu$, $\dot{\mathbf{r}}_c$ are evaluated in the $Oxyz$ system of coordinates.

By virtue of the identity [7]

$$\mathbf{v}_0 \cdot (\dot{\mathbf{r}}_c + \boldsymbol{\omega} \times \mathbf{r}_c) = d(\mathbf{r}_c \cdot \mathbf{v}_0) / dt - \mathbf{r}_c \cdot \mathbf{w}_0$$

where $\mathbf{w}_0$ is the absolute acceleration vector of the point O, we can rewrite (2.8) as follows:

$$T_a = T_r + \boldsymbol{\omega} \cdot \mathbf{G}_r + \boldsymbol{\omega} \cdot \boldsymbol{\theta}^\circ \cdot \boldsymbol{\omega} / 2 - M\mathbf{r}_c \cdot \mathbf{w}_0 + df / dt, \quad f(t, x) = M\mathbf{r}_c \cdot \mathbf{v}_0 \tag{2.9}$$

omitting the term $M\upsilon_0^2/2$, which does not affect the equations of motion. Thus, the generalized Lagrangian for the relative motion has the form of (2.2) with

$$\Lambda(t,x,\eta)=T_r(x,\eta)+\boldsymbol{\omega}(t)\cdot\mathbf{G}_r(x,\eta)+U^*(t,x)$$
$$U^*(t,x)=U(t,x)+\boldsymbol{\omega}(t)\cdot\boldsymbol{\theta}^\circ(x)\cdot\boldsymbol{\omega}(t)/2-M\mathbf{r}_c(x)\cdot\mathbf{w}_0(t) \tag{2.10}$$

where $U(t,x)$ is the force function of the active forces driving the system.

Consequently, in view of (2.10), the equations of relative motion of a holonomic system may be expressed either in the Poincaré form (2.1) or (2.3), or in the Chetayev form (2.6) or (2.7), which are equivalent. The results of [8] are special cases of this statement.

3. THE EQUATIONS OF MOTION IN NON-LINEAR QUASI-COORDINATES

It was shown in [3] that the Boltzmann–Hamel equations in linear quasi-coordinates are a special case of the generalized Poincaré equations, constructed using a closed system of infinitesimal virtual displacement operators; the canonical Chetayev form of equations in quasi-coordinates was also given.

Equations in non-linear quasi-coordinates were first derived by Hamel [9] from the central Lagrange equation, using transitivity equations. We will now derive those equations directly from the Lagrange or Maggi equations without using the transitivity equations, as in the derivation of the Poincaré equations in [4]; we will also compare the derived equations with the latter and convert them to canonical form.

We will first consider a holonomic system with Lagrange coordinates q_i $(i = 1,\dots,n)$ and arbitrary, generally non-linear, functions which are independent with respect to $\dot{q}_j$

$$\eta_i\equiv f_i(t,q,\dot{q}),\quad i=1,\dots n \tag{3.1}$$

such that $\det(\partial f_i/\partial\dot{q}_j)\neq 0$ $(i,j = 1,\dots,n)$. Solving (3.1), we obtain the expressions

$$\dot{q}_i\equiv F_i(t,q,\eta),\quad i=1,\dots,n \tag{3.2}$$

substitution of which into (3.1) yields identities, where (δ_{sr} is the Kronecker delta)

$$f_{si}F_{ir}=f_{ir}F_{si}=\delta_{sr} \tag{3.3}$$
$$f_{si}\equiv\partial f_s/\partial\dot{q}_i,\quad F_{ir}\equiv\partial F_i/\partial\eta_r,\quad i,j,r,s=1,\dots,n$$

Following Hamel, we put $\dot{\pi}_s=\eta_s$, where π_s are non-linear quasi-coordinates and η_s are quasi-velocities, or, following Poincaré, parameters of actual displacements; we define "partial derivatives with respect to the quasi-coordinates", and the inverse expressions and variations by

$$\frac{\partial}{\partial\pi_s}=F_{is}\frac{\partial}{\partial q_i},\quad \frac{\partial}{\partial q_i}=f_{si}\frac{\partial}{\partial\pi_s} \tag{3.4}$$

$$\delta q_i=F_{is}\delta\pi_s,\quad \delta\pi_s=f_{si}\delta q_i,\quad i,s=1,\dots,n \tag{3.5}$$

Multiplying the Lagrange equations by F_{is} and summing over all i, we obtain the Maggi equations

$$\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}-\frac{\partial L}{\partial q_i}\right)F_{is}=Q_s^*,\quad Q_s^*=Q_iF_{is};\quad i,s=1,\dots,n \tag{3.6}$$

Replacing the velocities $\dot{q}_i$ in the function $L(t,q,\dot{q})$ by their expressions (3.2), we obtain the generalized Lagrangian $L^*(t,q,\eta)$. Since

$$\frac{\partial L}{\partial\dot{q}_i}=\frac{\partial L^*}{\partial\eta_r}\frac{\partial f_r}{\partial\dot{q}_i},\quad \frac{\partial L}{\partial q_i}=\frac{\partial L^*}{\partial q_i}+\frac{\partial L^*}{\partial\eta_r}\frac{\partial f_r}{\partial q_i}$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}=\frac{d}{dt}\left(\frac{\partial L^*}{\partial\eta_r}\right)\frac{\partial f_r}{\partial\dot{q}_i}+\frac{\partial L^*}{\partial\eta_r}\frac{d}{dt}\frac{\partial f_r}{\partial\dot{q}_i};\quad i,r=1,\dots,n$$

it follows that Eqs (3.6), taking (3.3) and (3.4) into account, become the equations of motion of a holonomic system in non-linear quasi-coordinates

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s}+\frac{\partial L^*}{\partial \eta_r}\left(\frac{d}{dt}\frac{\partial f_r}{\partial \dot{q}_i}-\frac{\partial f_r}{\partial q_i}\right)F_{is}-\frac{\partial L^*}{\partial \pi_s}=Q_s^*;\quad r,s,i=1,\ldots,n \tag{3.7}$$

Using the equalities

$$\frac{\partial L^*}{\partial \eta_r}\left(\frac{d}{dt}\frac{\partial f_r}{\partial \dot{q}_i}-\frac{\partial f_r}{\partial q_i}\right)F_{is}=-\frac{\partial L^*}{\partial \eta_i}\left(\frac{d}{dt}\frac{\partial F_r}{\partial \eta_s}-\frac{\partial F_r}{\partial \pi_s}\right)f_{ir} \tag{3.8}$$

which follow from the transitivity equations, Hamel [9] transformed Eqs (3.7) to a second form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s}-\frac{\partial L^*}{\partial \eta_i}\left(\frac{d}{dt}\frac{\partial F_r}{\partial \eta_s}-\frac{\partial F_r}{\partial \pi_s}\right)f_{ir}-\frac{\partial L^*}{\partial \pi_s}=Q_s^*;\quad r,s,i=1,\ldots,n \tag{3.9}$$

Let us consider the variation of a function $\varphi(t,q)\in C^2$ over virtual displacements of system (3.5)

$$\delta\varphi=\frac{\partial \varphi}{\partial q_i}\delta q_i=\frac{\partial \varphi}{\partial q_i}F_{is}\delta\pi_s=\frac{\partial \varphi}{\partial \pi_s}\delta\pi_s=X_s\varphi\delta\pi_s$$

from which we obtain expressions for the virtual displacement operators

$$X_s\varphi\equiv\frac{\partial \varphi}{\partial \pi_s}=F_{is}\frac{\partial \varphi}{\partial q_i},\quad s=1,\ldots,n \tag{3.10}$$

with commutator

$$[X_r,X_s]\varphi\equiv X_rX_s\varphi-X_sX_r\varphi=c_{rs}^mX_m\varphi;\quad m,r,s=1,\ldots,n$$

where the structure coefficients

$$c_{rs}^m\equiv\left(F_{ir}\frac{\partial F_{js}}{\partial q_i}-F_{is}\frac{\partial F_{jr}}{\partial q_i}\right)f_{mj};\quad i,j=1,\ldots,n$$

generally depend not only on t and q_i but also on η_s on which the operators (3.1) themselves also depend. Thus, the system of operators (3.10) is a closed system, like the operators (1.10) of [4] in the generalized Poincaré equations.

Introducing the notation [10]

$$T_s^i\equiv\frac{d}{dt}\frac{\partial F_i}{\partial \eta_s}-\frac{\partial F_i}{\partial \pi_s},\quad W_s^r\equiv F_{is}\left(\frac{d}{dt}\frac{\partial f_r}{\partial \dot{q}_i}-\frac{\partial f_r}{\partial q_i}\right);\quad i,r,s=1,\ldots,n \tag{3.11}$$

for the Chaplygin and Voronets–Hamel coefficients, respectively, and taking (3.10) into account, we can write Eqs (3.7) and (3.9) in the more compact form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s}+\frac{\partial L^*}{\partial \eta_r}W_s^r-X_sL^*=Q_s^*;\quad i,r,s=1,\ldots,n \tag{3.12}$$

and

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s}-\frac{\partial L^*}{\partial \eta_i}f_{ir}T_s^r-X_sL^*=Q_s^*;\quad i,r,s=1,\ldots,n \tag{3.13}$$

analogous to the form of the Poincaré equations.

In the notation (3.11), Hamel's transitivity equations (13) and (13a) [9] take the following form [10]

$$\frac{d\delta\pi_s}{dt} - \delta\eta_s = W_r^s \delta\pi_r = -\frac{\partial\eta_s}{\partial\dot{q}_i} T_r^i \delta\pi_r; \quad i,r,s = 1,\ldots,n \tag{3.14}$$

analogous to that of the Poincaré equations (1.11) of [4], which link the parameters of the virtual and actual displacements.

It follows from the foregoing that, at least formally, the equations of motion in non-linear quasi-coordinates are analogous to the generalized Poincaré equations.

We will now reduce Eqs (3.12) and (3.13) to Chetayev canonical form. Assuming that $\|\partial^2 L^*/\partial\eta_r, \partial\eta_s\| \neq 0$ we replace the variables η_s and the function $L^*(t, q, \eta)$ by new variables y_s and a function $H^*(t, q, y)$ via the equalities

$$y_s = \frac{\partial L^*}{\partial\eta_s}, \quad H^*(t,q,y) = y_s\eta_s - L^*(t,q,\eta) \tag{3.15}$$

from which we obtain the following relations [2]

$$\eta_s = \frac{\partial H^*}{\partial y_s}, \quad X_s H^* = -X_s L^*, \quad s = 1,\ldots,n \tag{3.16}$$

Substituting (3.15) and (3.16) into Eqs (3.12) and (3.13) we convert the latter to the form of the canonical Chetayev equations

$$\frac{dy_s}{dt} + y_r W_s^r + X_s H^* = Q_s^*, \quad \eta_s = \frac{\partial H^*}{\partial y_s}; \quad i,r,s = 1,\ldots,n \tag{3.17}$$

$$\frac{dy_s}{dt} - y_i T_s^r f_{ir} + X_s H^* = Q_s^*, \quad \eta_s = \frac{\partial H^*}{\partial y_s}; \quad i,r,s = 1,\ldots,n \tag{3.18}$$

in the variables q_i and y_s.

In conclusion, let us consider a non-holonomic system with k degrees of freedom subject to non-integrable constraints of the form

$$\eta_j \equiv f_j(t,q,\dot{q}) = 0, \quad j = k+1,\ldots,n \tag{3.19}$$

We add to (3.19) arbitrary independent relations

$$\eta_s \equiv f_s(t,q,\dot{q}), \quad s = 1,\ldots,k \tag{3.20}$$

such that $\det(\partial f_i/\partial\dot{q}_j) \neq 0$ $(i, j = 1, \ldots, n)$. The virtual displacements of the system are defined by Chetayev's conditions

$$\frac{\partial f_j}{\partial\dot{q}_i}\delta q_i = 0, \quad i = 1,\ldots,n; \quad j = k+1,\ldots,n \tag{3.21}$$

Noting that under the constraints (3.19) $\delta\pi_j = 0$ $(j = k + 1, \ldots, n)$, we find that the virtual displacements, as before, satisfy equalities (3.5), but with $s = 1, \ldots, k$. The Maggi equations (3.6) for a non-holonomic system, $s = 1, \ldots, k$, as in the previous section of this paper, may be transformed to the form of (3.12) or (3.13), (3.17) or (3.18), where $s = 1, \ldots, k$; these are the equations of motion of the non-holonomic system in non-linear quasi-coordinates, which are identical with Hamel's equations (I) and (II) [9]. These equations must be complemented with the constraint equations (3.19). The transitivity equations (3.14) retain their form, if allowance is made for the fact that $\delta\pi_r = 0$ for $r = k + 1, \ldots, n$.

Note that Eqs (3.7) and (3.9), as well as Eqs (3.17) and (3.18), in the case of linear quasi-coordinates, take the form of generalized Poincaré and Chetayev equations [4] (as to the Poincaré equations see [11]).

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